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Elastic Scattering from a Spherical Body Eccentrically Located Within a Sphere: Theoretical Derivation

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Abstract

The scattered fields from a spherical body eccentrically located within a host sphere are found by a derivation that satisfies the boundary conditions at both interfaces. The source, which may be composed of any linear combination of *S* and *P* waves, is also arbitrarily located within the host sphere. The scattering system has applications in seismic scattering, since the scattering interaction between a scatterer and the Earth's surface is significant when the scatterer is located near the surface. This interaction can affect subsequent seismograms.

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1. Introduction

Scattering of elastic waves by spherical bodies is often studied in the context of seismic theory. The simplest system for which an analytical solution can readily be obtained for the entire scattered elastic wave field is that of a sphere in an infinite elastic medium (Ying and Truell, 1956; Einspruch, Witterholt, and Truell, 1960; Dubrovskiy and Morochnik, 1989). More complicated techniques must be employed to take into consideration the effects of multiple scattering (Wu, 1985; Wu and Aki, 1985) and irregularities in the geometry of scatterers (Waterman, 1969, 1976; Varatharajulu and Pao, 1976). When an inhomogeneity is located near an interface, the interaction between the inhomogeneity and the interface is not insignificant compared to the incident field, and must be considered when the scattered fields are calculated.

In this report, we derive the equations describing the scattered field from a spherical scattering body eccentrically located within a host sphere. The interface, in this case, is the host sphere (free) surface, which can be made large compared with the size of the scatterer. Such a system is of interest in seismology, since it can be used to describe the scattered fields due to inhomogeneities located near the surface of the Earth (host sphere). In order to solve this problem, we expand the relevant wave fields in terms of vector spherical harmonics and use the interface boundary conditions at both interfaces. In order to solve both sets of boundary conditions simultaneously, we translate the spherical vector harmonics representing the individual fields between the coordinate systems of the host and scattering spheres. This work is similar to previous works in which eccentric sources are embedded within a spherical host (Thompson, 1973; Glenn *et al*, 1985; Rial and Moran, 1986; Zhao and Harkrider, 1992); however, in addition to an eccentric source, we also include an eccentrically located scatterer. The notation used in this report is similar to that of a previous paper on light scattering from a host sphere containing an eccentric inclusion (Videen *et al*, 1995).

2. Theory

The geometry of the scattering system is shown in figure 1. A host sphere of radius a and elastic moduli λ and μ is centered on the x_1, y_1, z_1 coordinate system. A scattering sphere of radius a' and elastic moduli λ' and μ' is centered on the x_2, y_2, z_2 coordinate system, located at $(x_1 = 0, y_1 = 0, z_1 = d)$. A source is centered on the x_3, y_3, z_3 coordinate system, located at $(r_s = r_s, \theta_s = \theta_s, \varphi_s = 0)$ such that \hat{z}_3 is parallel to \hat{r}_s , and \hat{y}_1 is parallel to \hat{y}_3 . A time dependence of $\exp(-i\omega t)$ is implicit. Real seismic sources have a more complicated time dependence. For linear, homogeneous systems, we can solve the equations for a general time-dependent source using integral Fourier transformations:

$$f(x_j) = \int_{-\infty}^{\infty} F(k_j) \exp(-ix_j k_j) dk_j , \quad (1)$$

$$F(k_j) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x_j) \exp(ix_j k_j) dx_j . \quad (2)$$

In this report we solve for the amplitudes of the scattered fields at specific frequencies $F(k_j)$, which we can transform to find the time-dependent amplitudes using equation (1).

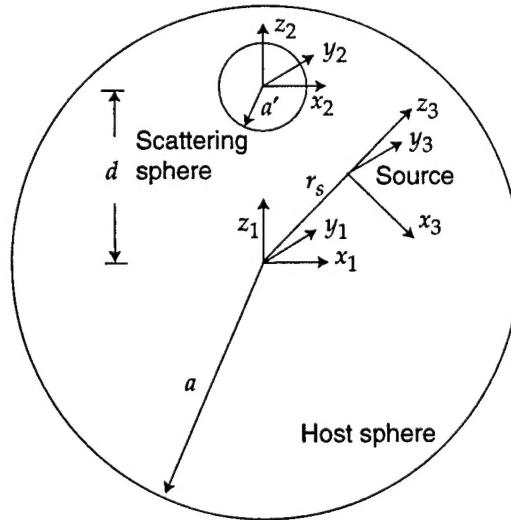


Figure 1. Geometry of scattering system showing coordinate systems of host (x_1, y_1, z_1) , scatterer (x_2, y_2, z_2) , and source (x_3, y_3, z_3) .

We reach the general solution by satisfying the boundary conditions on both spherical interfaces simultaneously. In doing so, we consider the fields incident on the interfaces of the two subsystems (host and scattering spheres) separately. We use the vector spherical harmonics (Hansen vectors), which have the following form in this derivation:

$$\begin{aligned} \mathbf{L}_{nm,j}^{(\rho)} &= \hat{r}_j \left[z_n^{(\rho)\prime}(kr_j) \tilde{P}_n^m(\cos \theta_j) e^{im\varphi_j} \right] + \\ &\quad \hat{\theta}_j \left[\frac{1}{kr_j} z_n^{(\rho)}(kr_j) \frac{d}{d\theta_j} \tilde{P}_n^m(\cos \theta_j) e^{im\varphi_j} \right] + \\ &\quad \hat{\varphi}_j \left[\frac{1}{kr_j} z_n^{(\rho)}(kr_j) \frac{im}{\sin \theta_j} \tilde{P}_n^m(\cos \theta_j) e^{im\varphi_j} \right], \end{aligned} \quad (3)$$

$$\begin{aligned} \mathbf{M}_{nm,j}^{(\rho)} &= \hat{\theta}_j \left[\frac{im}{\sin \theta_j} z_n^{(\rho)}(kr_j) \tilde{P}_n^m(\cos \theta_j) e^{im\varphi_j} \right] - \\ &\quad \hat{\varphi}_j \left[z_n^{(\rho)}(kr_j) \frac{d}{d\theta_j} \tilde{P}_n^m(\cos \theta_j) e^{im\varphi_j} \right], \end{aligned} \quad (4)$$

$$\begin{aligned} \mathbf{N}_{nm,j}^{(\rho)} &= \hat{r}_j \left[\frac{1}{kr_j} z_n^{(\rho)}(kr_j) n(n+1) \tilde{P}_n^m(\cos \theta_j) e^{im\varphi_j} \right] + \\ &\quad \hat{\theta}_j \left[\frac{1}{kr_j} \frac{d}{dr_j} (r_j z_n^{(\rho)}(kr_j)) \frac{d}{d\theta_j} \tilde{P}_n^m(\cos \theta_j) e^{im\varphi_j} \right] + \\ &\quad \hat{\varphi}_j \left[\frac{1}{kr_j} \frac{d}{dr_j} (r_j z_n^{(\rho)}(kr_j)) \frac{im}{\sin \theta_j} \tilde{P}_n^m(\cos \theta_j) e^{im\varphi_j} \right], \end{aligned} \quad (5)$$

where the prime denotes a derivative with respect to the argument, the index j corresponds to the coordinate system used ($j = 1, 2$), and $z_n^{(\rho)}(kr_j)$ are the spherical Bessel functions of the first, second, third, or fourth kind ($\rho = 1, 2, 3, 4$) (Ben-Menahem and Singh, 1981), and

$$\tilde{P}_n^m(\cos \theta_j) = \sqrt{\frac{(2n+1)(n-m)!}{2(n+m)!}} P_n^m(\cos \theta_j), \quad (6)$$

where $P_n^m(\cos \theta_j)$ are the associated Legendre polynomials.

It is convenient to consider the vector surface harmonics, \mathbf{P}_{nm} , \mathbf{B}_{nm} , and \mathbf{C}_{nm} when solving the boundary conditions. The vector surface harmonics

are defined in terms of the vector spherical harmonics:

$$\mathbf{L}_{nm,j}^{(\rho)} = \mathbf{P}_{nm,j} z_n^{(\rho)\prime}(kr_j) + \mathbf{B}_{nm,j} \frac{1}{kr_j} z_n^{(\rho)}(kr_j), \quad (7)$$

$$\mathbf{M}_{nm,j}^{(\rho)} = \mathbf{C}_{nm,j} z_n^{(\rho)}(kr_j), \quad (8)$$

$$\mathbf{N}_{nm,j}^{(\rho)} = \mathbf{P}_{nm,j} \frac{1}{kr_j} z_n^{(\rho)}(kr_j) n(n+1) + \mathbf{B}_{nm,j} \left[\frac{1}{kr_j} \frac{d}{dr_j} (r_j z_n^{(\rho)}(kr_j)) \right]. \quad (9)$$

We start the derivation by expanding the fields in terms of the vector spherical harmonics and satisfying the boundary conditions on the interface of the scattering sphere.

2.1 Scattering Sphere

First we examine the fields that strike the outer surface of the scattering sphere. We consider an arbitrary field incident on the system that can be expanded with the spherical Bessel functions of the first kind, $j_n(kr_2)$:

$$\begin{aligned} \mathbf{u}_{sou}^2 &= \sum_{n=0}^{\infty} \sum_{m=-n}^n a_{nm}^{(1)} \mathbf{L}_{nm,2}^{(1)} + a_{nm}^{(2)} \mathbf{M}_{nm,2}^{(1)} + a_{nm}^{(3)} \mathbf{N}_{nm,2}^{(1)} \\ &= \sum_{n=0}^{\infty} \sum_{m=-n}^n \mathbf{P}_{nm,2} \left[a_{nm}^{(1)} z_n^{(1)\prime}(k_\alpha r_2) + a_{nm}^{(3)} n(n+1) \frac{z_n^{(1)}(k_\beta r_2)}{k_\beta r_2} \right] \\ &\quad + \mathbf{B}_{nm,2} \left(a_{nm}^{(1)} \frac{z_n^{(1)}(k_\alpha r_2)}{k_\alpha r_2} + a_{nm}^{(3)} \left[\frac{z_n^{(1)}(k_\beta r_2)}{k_\beta r_2} + z_n^{(1)\prime}(k_\beta r_2) \right] \right) \\ &\quad + \mathbf{C}_{nm,2} a_{nm}^{(2)} z_n^{(1)}(k_\beta r_2) \\ &= \sum_{n=0}^{\infty} \sum_{m=-n}^n A_{nm}^{(1)}(k_\alpha r_2, k_\beta r_2) \mathbf{P}_{nm,2} + A_{nm}^{(2)}(k_\alpha r_2, k_\beta r_2) \mathbf{B}_{nm,2} + A_{nm}^{(3)}(k_\beta r_2) \mathbf{C}_{nm,2}, \end{aligned} \quad (10)$$

where k_α and k_β are the compression and shear wavenumbers external to the scattering sphere, respectively. Similarly, k'_α and k'_β are the compression and shear wavenumbers internal to the scattering sphere, respectively. The coefficients $a_{nm}^{(1)}$, $a_{nm}^{(2)}$, and $a_{nm}^{(3)}$ represent the field from the source. Similarly, the scattered field from the scattering sphere may be expanded with the spherical Bessel functions of the third kind, $h_n^{(1)}(kr_2)$:

$$\begin{aligned}
\mathbf{u}_{sca}^2 &= \sum_{n=0}^{\infty} \sum_{m=-n}^n b_{nm}^{(1)} \mathbf{L}_{nm,2}^{(3)} + b_{nm}^{(2)} \mathbf{M}_{nm,2}^{(3)} + b_{nm}^{(3)} \mathbf{N}_{nm,2}^{(3)} \\
&= \sum_{n=0}^{\infty} \sum_{m=-n}^n B_{nm}^{(1)}(k_{\alpha}r_2, k_{\beta}r_2) \mathbf{P}_{nm,2} + B_{nm}^{(2)}(k_{\alpha}r_2, k_{\beta}r_2) \mathbf{B}_{nm,2} + B_{nm}^{(3)}(k_{\beta}r_2) \mathbf{C}_{nm,2}.
\end{aligned} \tag{11}$$

This scattered field interacts with the host sphere (free) surface, which reflects incoming spherical waves onto the scattering sphere. These may be expanded with the spherical Bessel functions of the fourth kind, $h_n^{(2)}(kr_2)$:

$$\begin{aligned}
\mathbf{u}_{ref}^2 &= \sum_{n=0}^{\infty} \sum_{m=-n}^n c_{nm}^{(1)} \mathbf{L}_{nm,2}^{(4)} + c_{nm}^{(2)} \mathbf{M}_{nm,2}^{(4)} + c_{nm}^{(3)} \mathbf{N}_{nm,2}^{(4)} \\
&= \sum_{n=0}^{\infty} \sum_{m=-n}^n C_{nm}^{(1)}(k_{\alpha}r_2, k_{\beta}r_2) \mathbf{P}_{nm,2} + C_{nm}^{(2)}(k_{\alpha}r_2, k_{\beta}r_2) \mathbf{B}_{nm,2} + C_{nm}^{(3)}(k_{\beta}r_2) \mathbf{C}_{nm,2}.
\end{aligned} \tag{12}$$

The fields internal to the scattering sphere may be expanded into standing waves with spherical Bessel functions of the first kind:

$$\begin{aligned}
\mathbf{u}_{int}^2 &= \sum_{n=0}^{\infty} \sum_{m=-n}^n d_{nm}^{(1)} \mathbf{L}_{nm,2}^{(1)} + d_{nm}^{(2)} \mathbf{M}_{nm,2}^{(1)} + d_{nm}^{(3)} \mathbf{N}_{nm,2}^{(1)} \\
&= \sum_{n=0}^{\infty} \sum_{m=-n}^n D_{nm}^{(1)}(k'_{\alpha}r_2, k'_{\beta}r_2) \mathbf{P}_{nm,2} + D_{nm}^{(2)}(k'_{\alpha}r_2, k'_{\beta}r_2) \mathbf{B}_{nm,2} + D_{nm}^{(3)}(k'_{\beta}r_2) \mathbf{C}_{nm,2}.
\end{aligned} \tag{13}$$

With the expansions of the fields defined, we can now satisfy the boundary conditions at the interface of the scattering sphere. The stress vector across a spherical surface for a displacement field given by

$$\mathbf{u} = U(r) \mathbf{P}_{nm} + V(r) \mathbf{B}_{nm} + W(r) \mathbf{C}_{nm} \tag{14}$$

can be calculated as (Ben-Menahem and Singh, 1981)

$$\begin{aligned}
\mathbf{T}(\hat{r}) = & \hat{r} \left[(\lambda + 2\mu) \frac{dU}{dr} + 2\lambda \frac{U}{r} - n(n+1)\lambda \frac{V}{r} \right] \tilde{P}_n^m(\cos \theta) e^{im\varphi} \\
& + \hat{\theta}\mu \left[\left(\frac{dV}{dr} + \frac{U-V}{r} \right) \frac{\partial}{\partial \theta} + \left(\frac{dW}{dr} - \frac{W}{r} \right) \frac{1}{\sin \theta} \frac{\partial}{\partial \varphi} \right] \tilde{P}_n^m(\cos \theta) e^{im\varphi} \\
& + \hat{\varphi}\mu \left[\left(\frac{dV}{dr} + \frac{U-V}{r} \right) \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} - \left(\frac{dW}{dr} - \frac{W}{r} \right) \frac{\partial}{\partial \varphi} \right] \tilde{P}_n^m(\cos \theta) e^{im\varphi}.
\end{aligned} \tag{15}$$

In order for the displacement field and the stress tensor to be continuous, the following six boundary conditions must be satisfied at $r_2 = a'$:

$$A_{nm}^{(1)}(k_\alpha r_2, k_\beta r_2) + B_{nm}^{(1)}(k_\alpha r_2, k_\beta r_2) + C_{nm}^{(1)}(k_\alpha r_2, k_\beta r_2) = D_{nm}^{(1)}(k'_\alpha r_2, k'_\beta r_2), \tag{16}$$

$$A_{nm}^{(2)}(k_\alpha r_2, k_\beta r_2) + B_{nm}^{(2)}(k_\alpha r_2, k_\beta r_2) + C_{nm}^{(2)}(k_\alpha r_2, k_\beta r_2) = D_{nm}^{(2)}(k'_\alpha r_2, k'_\beta r_2), \tag{17}$$

$$A_{nm}^{(3)}(k_\beta r_2) + B_{nm}^{(3)}(k_\beta r_2) + C_{nm}^{(3)}(k_\beta r_2) = D_{nm}^{(3)}(k'_\beta r_2), \tag{18}$$

$$\begin{aligned}
& \left[(\lambda + 2\mu) \frac{d}{dr_2} + \frac{2\lambda}{a'} \right] \left[A_{nm}^{(1)}(k_\alpha r_2, k_\beta r_2) + B_{nm}^{(1)}(k_\alpha r_2, k_\beta r_2) + C_{nm}^{(1)}(k_\alpha r_2, k_\beta r_2) \right] - \\
& n(n+1) \frac{\lambda}{a'} \left[A_{nm}^{(2)}(k_\alpha r_2, k_\beta r_2) + B_{nm}^{(2)}(k_\alpha r_2, k_\beta r_2) + C_{nm}^{(2)}(k_\alpha r_2, k_\beta r_2) \right] = \\
& \left[(\lambda' + 2\mu') \frac{d}{dr_2} + \frac{2\lambda'}{a'} \right] D_{nm}^{(1)}(k'_\alpha r_2, k'_\beta r_2) - n(n+1) \frac{\lambda'}{a'} D_{nm}^{(2)}(k'_\alpha r_2, k'_\beta r_2),
\end{aligned} \tag{19}$$

$$\begin{aligned}
& a'\mu \frac{d}{dr_2} \left[A_{nm}^{(2)}(k_\alpha r_2, k_\beta r_2) + B_{nm}^{(2)}(k_\alpha r_2, k_\beta r_2) + C_{nm}^{(2)}(k_\alpha r_2, k_\beta r_2) \right] + \\
& \mu \left[A_{nm}^{(1)}(k_\alpha r_2, k_\beta r_2) + B_{nm}^{(1)}(k_\alpha r_2, k_\beta r_2) + C_{nm}^{(1)}(k_\alpha r_2, k_\beta r_2) \right] - \\
& \mu \left[A_{nm}^{(2)}(k_\alpha r_2, k_\beta r_2) + B_{nm}^{(2)}(k_\alpha r_2, k_\beta r_2) + C_{nm}^{(2)}(k_\alpha r_2, k_\beta r_2) \right] \\
= & \mu' \left\{ a' \frac{d}{dr_2} D_{nm}^{(2)}(k'_\alpha r_2, k'_\beta r_2) + \left[D_{nm}^{(1)}(k'_\alpha r_2, k'_\beta r_2) - D_{nm}^{(2)}(k'_\alpha r_2, k'_\beta r_2) \right] \right\},
\end{aligned} \tag{20}$$

$$\mu \left[a' \frac{d}{dr_2} - 1 \right] \left[A_{nm}^{(3)}(k_\beta r_2) + B_{nm}^{(3)}(k_\beta r_2) + C_{nm}^{(3)}(k_\beta r_2) \right] = \mu' \left[a' \frac{d}{dr_2} - 1 \right] D_{nm}^{(3)}(k'_\beta r_2). \quad (21)$$

We are interested primarily in the external fields of the scattering sphere. By substituting equations (16) to (18) into equations (20) to (21), we can eliminate the internal field coefficients and express the scattered external field coefficients in terms of the incoming field coefficients. Then, by expanding the vector surface harmonics in terms of the vector spherical harmonics (given in eq (10)), we can derive the following expressions for the vector spherical harmonic coefficients:

$$b_{nm}^{(1)} = a_{nm}^{(1)} Q_{nm}^{11} + a_{nm}^{(3)} Q_{nm}^{13} + c_{nm}^{(1)} R_{nm}^{11} + c_{nm}^{(3)} R_{nm}^{13}, \quad (22)$$

$$b_{nm}^{(2)} = a_{nm}^{(2)} Q_{nm}^{22} + c_{nm}^{(2)} R_{nm}^{22}, \quad (23)$$

and

$$b_{nm}^{(3)} = a_{nm}^{(1)} Q_{nm}^{31} + a_{nm}^{(3)} Q_{nm}^{33} + c_{nm}^{(1)} R_{nm}^{31} + c_{nm}^{(3)} R_{nm}^{33}. \quad (24)$$

Explicit expressions for Q_{nm}^{11} , Q_{nm}^{13} , R_{nm}^{11} , R_{nm}^{13} , Q_{nm}^{22} , R_{nm}^{22} , Q_{nm}^{31} , Q_{nm}^{33} , R_{nm}^{31} , and R_{nm}^{33} are given in appendix A.

2.2 Host Sphere

We next examine the fields in the host (Earth-centered) coordinate system. First we consider a source that can be expanded with spherical Bessel functions of the third kind, $h_n^{(1)}(kr_1)$:

$$\begin{aligned} \mathbf{u}_{sou}^1 &= \sum_{n=0}^{\infty} \sum_{m=-n}^n e_{nm}^{(1)} \mathbf{L}_{nm,1}^{(3)} + e_{nm}^{(2)} \mathbf{M}_{nm,1}^{(3)} + e_{nm}^{(3)} \mathbf{N}_{nm,1}^{(3)} \\ &= \sum_{n=0}^{\infty} \sum_{m=-n}^n E_{nm}^{(1)}(r_1) \mathbf{P}_{nm,1} + E_{nm}^{(2)}(r_1) \mathbf{B}_{nm,1} + E_{nm}^{(3)}(r_1) \mathbf{C}_{nm,1}. \end{aligned} \quad (25)$$

The scattered and incident fields from the scattering sphere may be expanded with the spherical Bessel functions of the third kind, $h_n^{(1)}(kr_1)$,

$$\begin{aligned} \mathbf{u}_{sca}^1 &= \sum_{n=0}^{\infty} \sum_{m=-n}^n f_{nm}^{(1)} \mathbf{L}_{nm,1}^{(3)} + f_{nm}^{(2)} \mathbf{M}_{nm,1}^{(3)} + f_{nm}^{(3)} \mathbf{N}_{nm,1}^{(3)} \\ &= \sum_{n=0}^{\infty} \sum_{m=-n}^n F_{nm}^{(1)}(r_1) \mathbf{P}_{nm,1} + F_{nm}^{(2)}(r_1) \mathbf{B}_{nm,1} + F_{nm}^{(3)}(r_1) \mathbf{C}_{nm,1}, \end{aligned} \quad (26)$$

and the spherical Bessel functions of the fourth kind, $h_n^{(2)}(kr_1)$,

$$\begin{aligned}\mathbf{u}_{ref}^1 &= \sum_{n=0}^{\infty} \sum_{m=-n}^n g_{nm}^{(1)} \mathbf{L}_{nm,1}^{(4)} + g_{nm}^{(2)} \mathbf{M}_{nm,1}^{(4)} + g_{nm}^{(3)} \mathbf{N}_{nm,1}^{(4)} \\ &= \sum_{n=0}^{\infty} \sum_{m=-n}^n G_{nm}^{(1)}(r_1) \mathbf{P}_{nm,1} + G_{nm}^{(2)}(r_1) \mathbf{B}_{nm,1} + G_{nm}^{(3)}(r_1) \mathbf{C}_{nm,1}.\end{aligned}\quad (27)$$

In appendix B, we provide expansions whereby relationships can be derived between the $a_{nm}^{(p)}$ and $e_{nm}^{(p)}$, $b_{nm}^{(p)}$ and $f_{nm}^{(p)}$, and $c_{nm}^{(p)}$ and $g_{nm}^{(p)}$ coefficients. The expansions (26) and (27) are valid in the region where $r_1 > d$: i.e., near the host sphere (free) surface. Hence, we need not be concerned with the pole that occurs at $r_1 = 0$ for the scattered and incident fields, since the expansion is not valid in this region. Since the stress vector across a free surface must be zero, the following boundary conditions apply at $r_1 = a$:

$$\begin{aligned}\left[(\lambda + 2\mu) \frac{d}{dr_1} + \frac{2\lambda}{a} \right] (E_{nm}^{(1)}(r_1) + F_{nm}^{(1)}(r_1) + G_{nm}^{(1)}(r_1)) = \\ n(n+1) \frac{\lambda}{a} (E_{nm}^{(2)}(r_1) + F_{nm}^{(2)}(r_1) + G_{nm}^{(2)}(r_1)),\end{aligned}\quad (28)$$

$$a \frac{d}{dr_1} (E_{nm}^{(2)}(r_1) + F_{nm}^{(2)}(r_1) + G_{nm}^{(2)}(r_1)) + \quad (29)$$

$$(E_{nm}^{(1)}(r_1) + F_{nm}^{(1)}(r_1) + G_{nm}^{(1)}(r_1)) - (E_{nm}^{(2)}(r_1) + F_{nm}^{(2)}(r_1) + G_{nm}^{(2)}(r_1)) = 0,$$

$$a \frac{d}{dr_1} (E_{nm}^{(3)}(r_1) + F_{nm}^{(3)}(r_1) + G_{nm}^{(3)}(r_1)) - E_{nm}^{(3)}(r_1) + F_{nm}^{(3)}(r_1) + G_{nm}^{(3)}(r_1) = 0. \quad (30)$$

As with the scattering-sphere coefficients, we express the coefficients in the host-sphere coordinate system in the following form:

$$g_{nm}^{(1)} = (e_{nm}^{(1)} + f_{nm}^{(1)}) S_{nm}^{11} + (e_{nm}^{(3)} + f_{nm}^{(3)}) S_{nm}^{13}, \quad (31)$$

$$g_{nm}^{(2)} = (e_{nm}^{(2)} + f_{nm}^{(2)}) S_{nm}^{22}, \quad (32)$$

and

$$g_{nm}^{(3)} = (e_{nm}^{(1)} + f_{nm}^{(1)}) S_{nm}^{31} + (e_{nm}^{(3)} + f_{nm}^{(3)}) S_{nm}^{33}. \quad (33)$$

In appendix A, we give explicit expressions for S_{nm}^{11} , S_{nm}^{13} , S_{nm}^{22} , S_{nm}^{31} , and S_{nm}^{33} .

2.3 Fields Interior to the Host Sphere (Earth)

The displacement fields given by equations (10) to (12) are equivalent to the displacement fields given by equations (25) to (27). By performing a translation of the spherical vector harmonics from the host coordinate system to the scattering-sphere coordinate system (using the results of app B), we can put equations (31) to (33) into the following form:

$$e_{nm}^{(1)} S_{nm}^{11} + e_{nm}^{(3)} S_{nm}^{13} = \sum_{n'=0}^{\infty} c_{n'm}^{(1)} C_{n',n}^{(m,1)} - b_{n'm}^{(1)} S_{nm}^{11} C_{n',n}^{(m,1)} - b_{n'm}^{(3)} S_{nm}^{13} A_{n',n}^{(m,1)} - b_{n'm}^{(2)} S_{nm}^{13} B_{n',n}^{(m,1)}, \quad (34)$$

$$S_{nm}^{22} e_{nm}^{(2)} = \sum_{n'=0}^{\infty} c_{n'm}^{(2)} A_{n',n}^{(m,1)} + c_{n'm}^{(3)} B_{n',n}^{(m,1)} - b_{n'm}^{(2)} S_{nm}^{22} A_{n',n}^{(m,1)} - b_{n'm}^{(3)} S_{nm}^{22} B_{n',n}^{(m,1)}, \quad (35)$$

and

$$e_{nm}^{(1)} S_{nm}^{31} + e_{nm}^{(3)} S_{nm}^{33} = \sum_{n'=0}^{\infty} c_{n'm}^{(3)} A_{n',n}^{(m,1)} + c_{n'm}^{(2)} B_{n',n}^{(m,1)} - b_{n'm}^{(1)} S_{nm}^{31} C_{n',n}^{(m,1)} - b_{n'm}^{(3)} S_{nm}^{33} A_{n',n}^{(m,1)} - b_{n'm}^{(2)} S_{nm}^{33} B_{n',n}^{(m,1)}. \quad (36)$$

Substituting equations (22) to (24) into these expressions yields

$$\begin{aligned} e_{nm}^{(1)} S_{nm}^{11} + e_{nm}^{(3)} S_{nm}^{13} &= - \sum_{n'=0}^{\infty} a_{n'm}^{(1)} (Q_{n'm}^{11} S_{nm}^{11} C_{n',n}^{(m,1)} + Q_{n'm}^{31} S_{nm}^{13} A_{n',n}^{(m,1)}) + \\ &\quad a_{n'm}^{(2)} Q_{n'm}^{22} S_{nm}^{13} B_{n',n}^{(m,1)} + \\ &\quad a_{n'm}^{(3)} (Q_{n'm}^{13} S_{nm}^{11} C_{n',n}^{(m,1)} + Q_{n'm}^{33} S_{nm}^{13} A_{n',n}^{(m,1)}) + \\ &\quad c_{n'm}^{(1)} (R_{n'm}^{11} S_{nm}^{11} C_{n',n}^{(m,1)} + R_{n'm}^{31} S_{nm}^{13} A_{n',n}^{(m,1)} - C_{n',n}^{(m,1)}) + \\ &\quad c_{n'm}^{(2)} R_{n'm}^{22} S_{nm}^{13} B_{n',n}^{(m,1)} + \\ &\quad c_{n'm}^{(3)} (R_{n'm}^{13} S_{nm}^{11} C_{n',n}^{(m,1)} + R_{n'm}^{33} S_{nm}^{13} A_{n',n}^{(m,1)}), \end{aligned} \quad (37)$$

$$\begin{aligned}
e_{nm}^{(2)} S_{nm}^{22} = & - \sum_{n'=0}^{\infty} a_{n'm}^{(1)} (Q_{n'm}^{31} S_{nm}^{22} B_{n',n}^{(m,1)}) + c_{n'm}^{(1)} (R_{n'm}^{31} S_{nm}^{22} B_{n',n}^{(m,1)}) + \\
& a_{n'm}^{(2)} (Q_{n'm}^{22} S_{nm}^{22} A_{n',n}^{(m,1)}) + c_{n'm}^{(2)} (R_{n'm}^{22} S_{nm}^{22} A_{n',n}^{(m,1)} - A_{n',n}^{(m,1)}) + \\
& a_{n'm}^{(3)} (Q_{n'm}^{33} S_{nm}^{22} B_{n',n}^{(m,1)}) + c_{n'm}^{(3)} (R_{n'm}^{33} S_{nm}^{22} B_{n',n}^{(m,1)} - B_{n',n}^{(m,1)}),
\end{aligned} \tag{38}$$

and

$$\begin{aligned}
e_{nm}^{(1)} S_{nm}^{31} + e_{nm}^{(3)} S_{nm}^{33} = & - \sum_{n'=0}^{\infty} a_{n'm}^{(1)} (Q_{n'm}^{11} S_{nm}^{31} C_{n',n}^{(m,1)} + Q_{n'm}^{31} S_{nm}^{33} A_{n',n}^{(m,1)}) + \\
& a_{n'm}^{(2)} (Q_{n'm}^{22} S_{nm}^{33} B_{n',n}^{(m,1)}) + \\
& a_{n'm}^{(3)} (Q_{n'm}^{13} S_{nm}^{31} C_{n',n}^{(m,1)} + Q_{n'm}^{33} S_{nm}^{33} A_{n',n}^{(m,1)}) + \\
& c_{n'm}^{(1)} (R_{n'm}^{11} S_{nm}^{31} C_{n',n}^{(m,1)} + R_{n'm}^{31} S_{nm}^{33} A_{n',n}^{(m,1)}) + \\
& c_{n'm}^{(2)} (R_{n'm}^{22} S_{nm}^{33} B_{n',n}^{(m,1)} - B_{n',n}^{(m,1)}) + \\
& c_{n'm}^{(3)} (R_{n'm}^{13} S_{nm}^{31} C_{n',n}^{(m,1)} + R_{n'm}^{33} S_{nm}^{33} A_{n',n}^{(m,1)} - A_{n',n}^{(m,1)}).
\end{aligned} \tag{39}$$

The $a_{nm}^{(p)}$ and $e_{nm}^{(p)}$ coefficients are related by equations (B-14) to (B-16) (app B). Since the $e_{nm}^{(p)}$ coefficients are the coefficients for the incident field, which is assumed to be known, the $a_{nm}^{(p)}$ can be easily determined, and equations (37) to (39) represent three sets of equations containing three sets of unknowns $c_{nm}^{(p)}$. Numerical results based on this analytic solution can be acquired following the same algorithms used in light scattering (Videen *et al*, 1995). Equations relating the various fields are given by equations (22) to (24) and (31) to (33); the coefficients used in these equations can be calculated from the equations given in appendix A. Computational results obtained from similar expressions in the field of light scattering have shown that the coefficients of the scattering harmonic terms tend to have a negligible contribution to the scattered fields for $n > ka + 4(ka)^{1/3} + 2$ (Wiscombe, 1980; Bohren and Huffman, 1983). It is likely that a similar truncation criterion holds for the elastic scatter. With the truncation of the summations given by equations (37) to (39), a solution can be found through matrix inversion. With the $a_{nm}^{(p)}$ and $c_{nm}^{(p)}$ known, $b_{nm}^{(p)}$ can be determined by equations (22) to (24). Hence, the entire displacement field within the host sphere can be found.

2.4 Incident Field

Although the solution derived in section 2.3 is completely general, in many applications we are interested in a specific case where a point source is located at some position within the host sphere ($r_1 = r_s$, $\theta_1 = \theta_s$, $\varphi_1 = 0$). We specify a source coordinate system (x_3, y_3, z_3) such that \hat{z}_3 is parallel to \hat{r}_s and \hat{y}_1 is parallel to \hat{y}_3 , as shown in figure 1. The derivation is valid only when the source is closer to the center of the host sphere than is the scattering-sphere center ($r_s < d$). The limitation of the validity is a result of the expansion that we use in translating the vector spherical harmonics. If the source is farther from the center of the host sphere than the scattering sphere ($r_s > d$), then we would need to use a different set of translation coefficients, and the derivation would be slightly different (Stratton, 1941). The source can be expanded as

$$\mathbf{u}_{\text{sou}}^3 = \sum_{n=0}^{\infty} \sum_{m=-n}^n h_{nm}^{(1)} \mathbf{L}_{nm,3}^{(3)} + h_{nm}^{(2)} \mathbf{M}_{nm,3}^{(3)} + h_{nm}^{(3)} \mathbf{N}_{nm,3}^{(3)} . \quad (40)$$

This field can be expressed in a coordinate system whose origin is the same as the host (Earth) coordinate system, but whose axes are parallel to the source coordinate system by a translation of a distance r_s in the $-\hat{z}_3$ direction. The source in this (x_4, y_4, z_4) is

$$\mathbf{u}_{\text{sou}}^4 = \sum_{n=0}^{\infty} \sum_{m=-n}^n i_{nm}^{(1)} \mathbf{L}_{nm,4}^{(3)} + i_{nm}^{(2)} \mathbf{M}_{nm,4}^{(3)} + i_{nm}^{(3)} \mathbf{N}_{nm,4}^{(3)} . \quad (41)$$

We can rotate this vector field using the results of appendix B to give the coefficients of equation (25).

We provide formulae by which the incident field coefficients can be derived for two cases. These represent simple, but solvable sources. The incident field coefficients from more complicated sources can be substituted for the coefficients given in this section. In the first case, we consider the field due to an explosion. In this case, the source pressure wave is not angularly dependent, and we can express it as $-p_o \mathbf{L}_{00,3}^{(3)}$, or as

$$h_{nm}^{(1)} = h_{nm}^{(1)p} = -p_o \delta_{n,0} \delta_{m,0} , \quad (42)$$

$$h_{nm}^{(2)} = h_{nm}^{(2)p} = 0 , \quad (43)$$

and

$$h_{nm}^{(3)} = h_{nm}^{(3)p} = 0 . \quad (44)$$

In the second case, we consider a source that emits shear waves only, whose shearing stress $\tau_{r\varphi} = \tau_o \sin \theta_3$ is applied at the inner surface of a spherical cavity centered on the source. In this case the source coefficients are of the form

$$h_{nm}^{(1)} = h_{nm}^{(1)s} = 0 , \quad (45)$$

$$h_{nm}^{(2)} = h_{nm}^{(2)s} = \tau_o \delta_{n,1} \delta_{m,0} , \quad (46)$$

and

$$h_{nm}^{(3)} = h_{nm}^{(3)s} = 0 . \quad (47)$$

If the shear is in another direction, we can use the vector rotation results given in appendix B. The final expressions for the source coefficients in the host sphere coordinate system are given by equations (B-31) to (B-33).

3. Implications

In this report, we derive equations describing the scattered fields from a spherical scatterer located eccentrically within a host sphere. The source is also arbitrarily located within the host sphere, and can be chosen to simulate a *P* wave, *S* wave, or any linear combination of the two. We can use this theory in seismology to predict the scattering interaction between a scattering body and the Earth's surface.

The scattered fields are complete, and exact solutions for the elastic wave-field produced by the spherical scatterer are expressed. Because the solution presented here is almost entirely analytical, it avoids the need for much of the computationally intensive algorithms, such as finite-difference schemes, that would normally be required to solve this problem. Also, because we have made very few assumptions concerning the boundary conditions, the analytical solution given here should be well behaved.

This theory provides a basis that should prove useful in understanding and predicting seismic scattering. Using techniques developed in electromagnetic scattering (Videen *et al.*, 1995), it should be possible to extend this theory to describe the scattering from an arbitrary body rather than a spherical one. Hence, this theory is an important first step in demonstrating a method for analytically deriving scattered elastic wave fields from anomalies such as magma chambers or subducting slabs near the Earth's surface.

Appendix A. Boundary Coefficients

We can find the coefficients that satisfy the boundary conditions at the scattering sphere outer surface ($r_2 = a'$) by first eliminating the dependence on the internal field coefficients. Performing some algebraic manipulation and taking advantage of the differential equation defining spherical Bessel functions,

$$r^2 z_n''(r) + 2rz_n'(r) + [r^2 - n(n+1)] z_n(r) = 0, \quad (\text{A-1})$$

we can put the coefficients that satisfy the boundary conditions on the scattering sphere into the following form:

$$Q_{nm}^{11} = -\frac{\mathcal{A}_1^1 \mathcal{B}_2^3 - \mathcal{A}_2^1 \mathcal{B}_1^3}{\mathcal{B}_1^1 \mathcal{B}_2^3 - \mathcal{B}_2^1 \mathcal{B}_1^3}, \quad (\text{A-2})$$

$$Q_{nm}^{13} = -\frac{\mathcal{A}_1^3 \mathcal{B}_2^3 - \mathcal{A}_2^3 \mathcal{B}_1^3}{\mathcal{B}_1^1 \mathcal{B}_2^3 - \mathcal{B}_2^1 \mathcal{B}_1^3}, \quad (\text{A-3})$$

$$Q_{nm}^{22} = -\frac{J_n(1,1)}{J_n(3,1)}, \quad (\text{A-4})$$

$$Q_{nm}^{31} = \frac{\mathcal{A}_1^1 \mathcal{B}_2^1 - \mathcal{A}_2^1 \mathcal{B}_1^1}{\mathcal{B}_1^1 \mathcal{B}_2^3 - \mathcal{B}_2^1 \mathcal{B}_1^3}, \quad (\text{A-5})$$

$$Q_{nm}^{33} = \frac{\mathcal{A}_1^3 \mathcal{B}_2^1 - \mathcal{A}_2^3 \mathcal{B}_1^1}{\mathcal{B}_1^1 \mathcal{B}_2^3 - \mathcal{B}_2^1 \mathcal{B}_1^3}, \quad (\text{A-6})$$

and

$$R_{nm}^{11} = -\frac{\mathcal{C}_1^1 \mathcal{B}_2^3 - \mathcal{C}_2^1 \mathcal{B}_1^3}{\mathcal{B}_1^1 \mathcal{B}_2^3 - \mathcal{B}_2^1 \mathcal{B}_1^3}, \quad (\text{A-7})$$

$$R_{nm}^{13} = -\frac{\mathcal{C}_1^3 \mathcal{B}_2^3 - \mathcal{C}_2^3 \mathcal{B}_1^3}{\mathcal{B}_1^1 \mathcal{B}_2^3 - \mathcal{B}_2^1 \mathcal{B}_1^3}, \quad (\text{A-8})$$

$$R_{nm}^{22} = -\frac{J_n(4,1)}{J_n(3,1)}, \quad (\text{A-9})$$

$$R_{nm}^{31} = \frac{\mathcal{C}_1^1 \mathcal{B}_2^1 - \mathcal{C}_2^1 \mathcal{B}_1^1}{\mathcal{B}_1^1 \mathcal{B}_2^3 - \mathcal{B}_2^1 \mathcal{B}_1^3}, \quad (\text{A-10})$$

$$R_{nm}^{33} = \frac{\mathcal{C}_1^3 \mathcal{B}_2^1 - \mathcal{C}_2^3 \mathcal{B}_1^1}{\mathcal{B}_1^1 \mathcal{B}_2^3 - \mathcal{B}_2^1 \mathcal{B}_1^3}, \quad (\text{A-11})$$

where

$$\begin{aligned} \mathcal{A}_1^n &= (A_1^n D_2^3 - A_2^n D_1^3) (D_3^1 D_4^3 - D_4^1 D_3^3) \\ &\quad - (A_3^n D_4^3 - A_4^n D_3^3) (D_1^1 D_2^3 - D_2^1 D_1^3) \end{aligned} \quad (\text{A-12})$$

$$\begin{aligned} \mathcal{A}_2^n &= (A_1^n D_2^3 - A_2^n D_1^3) (D_2^1 D_3^3 - D_3^1 D_2^3) \\ &\quad - (A_2^n D_3^3 - A_3^n D_2^3) (D_1^1 D_2^3 - D_2^1 D_1^3) \end{aligned} \quad (\text{A-13})$$

$$\begin{aligned} \mathcal{B}_1^n &= (B_1^n D_2^3 - B_2^n D_1^3) (D_3^1 D_4^3 - D_4^1 D_3^3) \\ &\quad - (B_3^n D_4^3 - B_4^n D_3^3) (D_1^1 D_2^3 - D_2^1 D_1^3) \end{aligned} \quad (\text{A-14})$$

$$\begin{aligned} \mathcal{B}_2^n &= (B_1^n D_2^3 - B_2^n D_1^3) (D_2^1 D_3^3 - D_3^1 D_2^3) \\ &\quad - (B_2^n D_3^3 - B_3^n D_2^3) (D_1^1 D_2^3 - D_2^1 D_1^3) \end{aligned} \quad (\text{A-15})$$

$$\begin{aligned} \mathcal{C}_1^n &= (C_1^n D_2^3 - C_2^n D_1^3) (D_3^1 D_4^3 - D_4^1 D_3^3) \\ &\quad - (C_3^n D_4^3 - C_4^n D_3^3) (D_1^1 D_2^3 - D_2^1 D_1^3) \end{aligned} \quad (\text{A-16})$$

$$\begin{aligned} \mathcal{C}_2^n &= (C_1^n D_2^3 - C_2^n D_1^3) (D_2^1 D_3^3 - D_3^1 D_2^3) \\ &\quad - (C_2^n D_3^3 - C_3^n D_2^3) (D_1^1 D_2^3 - D_2^1 D_1^3) \end{aligned} \quad (\text{A-17})$$

$$A_j^1 = H_{2j-1}(1, k_\alpha, \lambda, \mu, a'), \quad (\text{A-18})$$

$$A_j^3 = H_{2j}(1, k_\beta, \lambda, \mu, a'), \quad (\text{A-19})$$

$$B_j^1 = H_{2j-1}(3, k_\alpha, \lambda, \mu, a'), \quad (\text{A-20})$$

$$B_j^3 = H_{2j}(3, k_\beta, \lambda, \mu, a'), \quad (\text{A-21})$$

$$C_j^1 = H_{2j-1}(4, k_\alpha, \lambda, \mu, a'), \quad (\text{A-22})$$

$$C_j^3 = H_{2j}(4, k_\beta, \lambda, \mu, a'), \quad (\text{A-23})$$

$$D_j^1 = H_{2j-1}(1, k'_\alpha, \lambda', \mu', a'), \quad (\text{A-24})$$

$$D_j^3 = H_{2j}(1, k'_\beta, \lambda', \mu', a'), \quad (\text{A-25})$$

$$J_n(i, j) = H_n(i, k_\beta, \lambda, \mu, a') - \frac{z_n^{(i)}}{z_n^{(j)}} H_n(j, k'_\beta, \lambda', \mu', a'),$$

and

$$H_1(p, k, \lambda, \mu, a) = 2\mu \left[z_n^{(p)\prime}(ka) - \frac{z_n^{(p)}(ka)}{ka} \right], \quad (\text{A-26})$$

$$H_2(p, k, \lambda, \mu, a) = 2\mu(n^2 + n - 1) \frac{z_n^{(p)}(ka)}{ka} - 2\mu z_n^{(p)\prime}(ka) - \mu k a z_n^{(p)}(ka), \quad (\text{A-27})$$

$$H_3(p, k, \lambda, \mu, a) = \left[\frac{2\mu n(n+1)}{ka} - ka(\lambda + 2\mu) \right] z_n^{(p)}(ka) - 4\mu z_n^{(p)\prime}(ka),$$

$$H_4(p, k, \lambda, \mu, a) = \mu n(n+1) H_1(p, k, a). \quad (\text{A-28})$$

The coefficients that satisfy the boundary conditions on the host-sphere outer surface ($r_1 = a$) are

$$S_{nm}^{11} = \frac{H_3(3, k_\alpha, \lambda, \mu, a) H_2(4, k_\beta, \lambda, \mu, a) - H_1(3, k_\alpha, \lambda, \mu, a) H_4(4, k_\beta, \lambda, \mu, a)}{H_1(4, k_\alpha, \lambda, \mu, a) H_4(4, k_\beta, \lambda, \mu, a) - H_3(4, k_\alpha, \lambda, \mu, a) H_2(4, k_\beta, \lambda, \mu, a)} \quad (\text{A-29})$$

$$S_{nm}^{13} = \frac{H_4(3, k_\beta, \lambda, \mu, a) H_2(4, k_\beta, \lambda, \mu, a) - H_2(3, k_\beta, \lambda, \mu, a) H_4(4, k_\beta, \lambda, \mu, a)}{H_1(4, k_\alpha, \lambda, \mu, a) H_4(4, k_\beta, \lambda, \mu, a) - H_3(4, k_\alpha, \lambda, \mu, a) H_2(4, k_\beta, \lambda, \mu, a)} \quad (\text{A-30})$$

$$S_{nm}^{22} = -\frac{H_1(3, k_\beta, \lambda, \mu, a)}{H_1(4, k_\beta, \lambda, \mu, a)} \quad (\text{A-31})$$

$$S_{nm}^{31} = \frac{H_3(3, k_\alpha, \lambda, \mu, a) H_1(4, k_\alpha, \lambda, \mu, a) - H_1(3, k_\alpha, \lambda, \mu, a) H_3(4, k_\alpha, \lambda, \mu, a)}{H_3(4, k_\alpha, \lambda, \mu, a) H_2(4, k_\beta, \lambda, \mu, a) - H_1(4, k_\alpha, \lambda, \mu, a) H_4(4, k_\beta, \lambda, \mu, a)} \quad (\text{A-32})$$

$$S_{nm}^{33} = \frac{H_4(3, k_\beta, \lambda, \mu, a) H_1(4, k_\alpha, \lambda, \mu, a) - H_2(3, k_\beta, \lambda, \mu, a) H_3(4, k_\alpha, \lambda, \mu, a)}{H_3(4, k_\alpha, \lambda, \mu, a) H_2(4, k_\beta, \lambda, \mu, a) - H_1(4, k_\alpha, \lambda, \mu, a) H_4(4, k_\beta, \lambda, \mu, a)} \quad (\text{A-33})$$

Appendix B. Vector Translations

Stein (1961) and Cruzan (1962) derived translation addition theorems for vector spherical wave functions that can be used to express the displacement fields in one coordinate system in terms of displacement fields in another coordinate system. These results are also given by Ben-Menaem and Singh (1981). We use their results to find relationships between the coefficients $a_{nm}^{(p)}$ and $e_{nm}^{(p)}$, $b_{nm}^{(p)}$ and $f_{nm}^{(p)}$, and $c_{nm}^{(p)}$ and $g_{nm}^{(p)}$. For a translation along the z -axis with no rotation, the vector spherical harmonics are related by

$$\mathbf{L}_{nm,2}^{(p)} = \sum_{n'=0}^{\infty} C_{n,n'}^{(m,i)} \mathbf{L}_{n'm,1}^{(q)}, \quad (\text{B-1})$$

$$\mathbf{M}_{nm,2}^{(p)} = \sum_{n'=0}^{\infty} A_{n,n'}^{(m,i)} \mathbf{M}_{n'm,1}^{(q)} + B_{n,n'}^{(m,i)} \mathbf{N}_{n'm,1}^{(q)}, \quad (\text{B-2})$$

$$\mathbf{N}_{nm,2}^{(p)} = \sum_{n'=0}^{\infty} B_{n,n'}^{(m,i)} \mathbf{M}_{n'm,1}^{(q)} + A_{n,n'}^{(m,i)} \mathbf{N}_{n'm,1}^{(q)}. \quad (\text{B-3})$$

If $p = 3$ or 4 , then $i = 1$ in the region where $r_1 < |d|$, and $i = p$ in the region where $r_i > |d|$. Recurrence relations for the scalar translation coefficients $C_{n,n'}^{(m,i)}$ were derived by Bobbert and Vlieger (1986), which greatly facilitate calculations. The results are

$$C_{0,n'}^{(0,i)} = \sqrt{2n'+1} z_{n'}^{(i)}(kd), \quad (\text{B-4})$$

$$C_{-1,n'}^{(0,i)} = -\sqrt{2n'+1} z_{n'}^{(i)}(kd), \quad (\text{B-5})$$

$$\begin{aligned} C_{n+1,n'}^{(0,i)} &= \frac{1}{(n+1)} \sqrt{\frac{2n+3}{2n'+1}} n' \sqrt{\frac{2n+1}{2n'-1}} C_{n,n'-1}^{(0,i)} + \\ &\quad n \sqrt{\frac{2n'+1}{2n-1}} C_{n-1,n'}^{(0,i)} - (n'+1) \sqrt{\frac{2n+1}{2n'+3}} C_{n,n'+1}^{(0,i)}, \end{aligned} \quad (\text{B-6})$$

$$\sqrt{(n-m+1)(n+m)(2n'+1)} C_{n,n'}^{(m,i)} = \sqrt{(n'-m+1)(n'+m)(2n'+1)} C_{n,n'}^{(m-1,i)} - \quad (\text{B-7})$$

$$kd \sqrt{\frac{(n'-m+2)(n'-m+1)}{(2n'+3)}} C_{n,n'+1}^{(m-1,i)} - kd \sqrt{\frac{(n'+m)(n'+m-1)}{(2n'-1)}} C_{n,n'-1}^{(m-1,i)}.$$

From the scalar translation coefficients, the vector translation coefficients can be easily derived:

$$A_{n,n'}^{(m,i)} = C_{n,n'}^{(m,i)} - \frac{kd}{n'+1} \sqrt{\frac{(n'-m+1)(n'+m+1)}{(2n'+1)(2n'+3)}} C_{n,n'+1}^{(m,i)} \quad (\text{B-8})$$

$$- \frac{kd}{n'} \sqrt{\frac{(n'-m)(n'+m)}{(2n'+1)(2n'-1)}} C_{n,n'-1}^{(m,i)},$$

$$B_{n,n'}^{(m,i)} = \frac{-imkd}{n'(n'+1)} C_{n,n'}^{(m,i)}. \quad (\text{B-9})$$

From these equations, we see that

$$A_{n,n'}^{(m,p,q)} = A_{n,n'}^{(-m,p,q)}, \quad (\text{B-10})$$

$$B_{n,n'}^{(m,p,q)} = B_{n,n'}^{(-m,p,q)}, \quad (\text{B-11})$$

$$C_{n,n'}^{(m,p,q)} = C_{n,n'}^{(-m,p,q)}. \quad (\text{B-12})$$

Since $\mathbf{u}_{sou}^1 = \mathbf{u}_{sou}^2$, $\mathbf{u}_{sca}^1 = \mathbf{u}_{sca}^2$, and $\mathbf{u}_{int}^1 = \mathbf{u}_{int}^2$, we can derive expressions relating the displacement coefficients using equations (B-1) to (B-3):

$$a_{nm}^{(1)} = \sum_{n'=0}^{\infty} e_{n'm}^{(1)} C_{n',n}^{(m,3)}(-k_{\beta}d), \quad (\text{B-13})$$

$$a_{nm}^{(2)} = \sum_{n'=0}^{\infty} e_{n'm}^{(2)} A_{n',n}^{(m,3)}(-k_{\beta}d) + e_{n'm}^{(3)} B_{n',n}^{(m,3)}(-k_{\beta}d), \quad (\text{B-14})$$

$$a_{nm}^{(3)} = \sum_{n'=0}^{\infty} e_{n'm}^{(3)} A_{n',n}^{(m,3)}(-k_{\beta}d) + e_{n'm}^{(2)} B_{n',n}^{(m,3)}(-k_{\beta}d) \quad (\text{B-15})$$

$$f_{nm}^{(1)} = \sum_{n'=0}^{\infty} b_{n'm}^{(1)} C_{n',n}^{(m,1)}(k_{\alpha}d), \quad (\text{B-16})$$

$$f_{nm}^{(2)} = \sum_{n'=0}^{\infty} b_{n'm}^{(2)} A_{n',n}^{(m,1)}(k_{\beta}d) + b_{n'm}^{(3)} B_{n',n}^{(m,1)}(k_{\beta}d), \quad (\text{B-17})$$

$$f_{nm}^{(3)} = \sum_{n'=0}^{\infty} b_{n'm}^{(3)} A_{n',n}^{(m,1)}(k_{\beta}d) + b_{n'm}^{(2)} B_{n',n}^{(m,1)}(k_{\beta}d), \quad (\text{B-18})$$

$$g_{nm}^{(1)} = \sum_{n'=0}^{\infty} c_{n'm}^{(1)} C_{n',n}^{(m,1)}(k_{\alpha}d), \quad (\text{B-19})$$

$$g_{nm}^{(2)} = \sum_{n'=0}^{\infty} c_{n'm}^{(2)} A_{n',n}^{(m,1)}(k_\beta d) + c_{n'm}^{(3)} B_{n',n}^{(m,1)}(k_\beta d), \quad (\text{B-20})$$

$$g_{nm}^{(3)} = \sum_{n'=0}^{\infty} c_{n'm}^{(3)} A_{n',n}^{(m,1)}(k_\beta d) + c_{n'm}^{(2)} B_{n',n}^{(m,1)}(k_\beta d). \quad (\text{B-21})$$

Finally, we note that the scalar translation coefficients $C_{n,n'}^{(m,i)}$ used in deriving the vector translation coefficients $A_{n,n'}^{(m,i)}$ and $B_{n,n'}^{(m,i)}$ are, in general, different from the vector translation coefficients $C_{n,n'}^{(m,i)}$ used in the translation of the P waves ($\mathbf{L}_{mn}^{(p)}$), since in general the P and S waves have different spatial frequencies. In addition, the translations for the source coefficients $a_{nm}^{(p)}$ and $e_{nm}^{(p)}$ are in opposite directions.

The translation from the source coordinate system to the rotated host coordinate system is similar:

$$i_{nm}^{(1)} = \sum_{n'=0}^{\infty} h_{n'm}^{(1)} C_{n',n}^{(m,1)}(k_\alpha r_s), \quad (\text{B-22})$$

$$i_{nm}^{(2)} = \sum_{n'=0}^{\infty} h_{n'm}^{(2)} A_{n',n}^{(m,1)}(k_\beta r_s) + h_{n'm}^{(3)} B_{n',n}^{(m,1)}(k_\beta r_s), \quad (\text{B-23})$$

$$i_{nm}^{(3)} = \sum_{n'=0}^{\infty} h_{n'm}^{(3)} A_{n',n}^{(m,1)}(k_\beta r_s) + h_{n'm}^{(2)} B_{n',n}^{(m,1)}(k_\beta r_s). \quad (\text{B-24})$$

Using the results of Stein (1960), we present the coefficients in a rotated coordinate system. The vector harmonics can be expanded as

$$\mathbf{L}_{nm,j}^{(p)} = \sum_{m'=0}^{\infty} D_{m'}^{(n,m)} \mathbf{L}_{nm',i}^{(p)}, \quad (\text{B-25})$$

$$\mathbf{M}_{nm,j}^{(p)} = \sum_{m'=0}^{\infty} D_{m'}^{(n,m)} \mathbf{M}_{nm',i}^{(q)}, \quad (\text{B-26})$$

$$\mathbf{N}_{nm,j}^{(p)} = \sum_{m'=0}^{\infty} D_{m'}^{(n,m)} \mathbf{N}_{nm',i}^{(q)}, \quad (\text{B-27})$$

where

$$D_{m'}^{(n,m)} = \exp [i(m'\alpha + m\gamma)] \left[\frac{(n+m')! (n-m')!}{(n+m)! (n-m)!} \right]^{1/2} \quad (B-28)$$

$$\times \sum_{\sigma} \binom{n+m}{n-m'-\sigma} \binom{n-m}{\sigma} (-1)^{n+m-\sigma} [\cos(\beta/2)]^{2\sigma+m'+m} \\ [\sin(\beta/2)]^{2n-2\sigma-m'-m} \quad (B-29)$$

and α , β , and γ are Euler angles using the convention of Edmonds (1957). The coefficients in the rotated coordinate system, $a_{nm,i}$, can be expressed in the unrotated coordinate system, $a_{nm,j}$, as

$$a_{nm,j} = \sum_{m'=0}^{\infty} D_{m'}^{(n,m)} a_{nm',i} . \quad (B-30)$$

Therefore, the source coefficients in the host coordinate system for the point source can be expressed as

$$e_{nm}^{(1)} = \sum_{m'=0}^{\infty} D_{m'}^{(n,m)} \sum_{n'=0}^{\infty} h_{n'm'}^{(1)} C_{n',n}^{(m',1)}(k_{\alpha} r_s) , \quad (B-31)$$

$$e_{nm}^{(2)} = \sum_{m'=0}^{\infty} D_{m'}^{(n,m)} \sum_{n'=0}^{\infty} h_{n'm'}^{(2)} A_{n',n}^{(m',1)}(k_{\beta} r_s) + h_{n'm'}^{(3)} B_{n',n}^{(m',1)}(k_{\beta} r_s) , \quad (B-32)$$

$$e_{nm}^{(3)} = \sum_{m'=0}^{\infty} D_{m'}^{(n,m)} \sum_{n'=0}^{\infty} h_{n'm'}^{(3)} A_{n',n}^{(m',1)}(k_{\beta} r_s) + h_{n'm'}^{(2)} B_{n',n}^{(m',1)}(k_{\beta} r_s) . \quad (B-33)$$

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